U-1

Elastic Constant.

The generalised Hooke's Law in contracted notation is,

$$\sigma_i = c_{ij}\, \epsilon_j \quad , \quad i, j = 1, 2 \ldots 6$$

$\Downarrow$

stiffness matrix / stiffness tensor.

By varying i, j we get 36 $\underbrace{\text{independent elements}}$.

$\Downarrow$

Elastic constants.

The reduction of elastic constants are based on the symmetry.

The strain energy density function W, is given as,

$$W = \frac{1}{2} c_{ij}\, \epsilon_i\, \epsilon_j \qquad \text{—— (1)}$$

The material with existence of W is called as hyper elastic materials.

W can also be written as,

$$W = \frac{1}{2} c_{ji}\, \epsilon_j\, \epsilon_i$$

$$= \frac{1}{2} c_{ji}\, \epsilon_i\, \epsilon_j \qquad \text{—— (2)}$$

(1) & (2) $\Rightarrow$ $\boxed{c_{ij} = c_{ji}}$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} = \sigma_4 \\ \tau_{31} = \sigma_5 \\ \tau_{12} = \sigma_6 \end{Bmatrix} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix}}_{\Downarrow \text{ 36 elastic constants.}} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}$$

$\Downarrow$ Tensor notation $\qquad$ $\Downarrow$ contracted notation

From $C_{ij} = C_{ji}$ i.e., $C_{12} = C_{21}$; $C_{31} = C_{13}$ ....

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}$$

$\Downarrow$

Symmetry.

ie, $C_{ij} = C_{ji} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{Symmetry} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$

$\Downarrow$

21 elastic constants.

The material with 21 independent elastic constants — Anisotropic materials.

We know,

$$W = \frac{1}{2} C_{ij} \epsilon_j \epsilon_i$$

ie, if $i=1$ and $j=2$

then, $W = \frac{1}{2} C_{12} \epsilon_1 \epsilon_2$

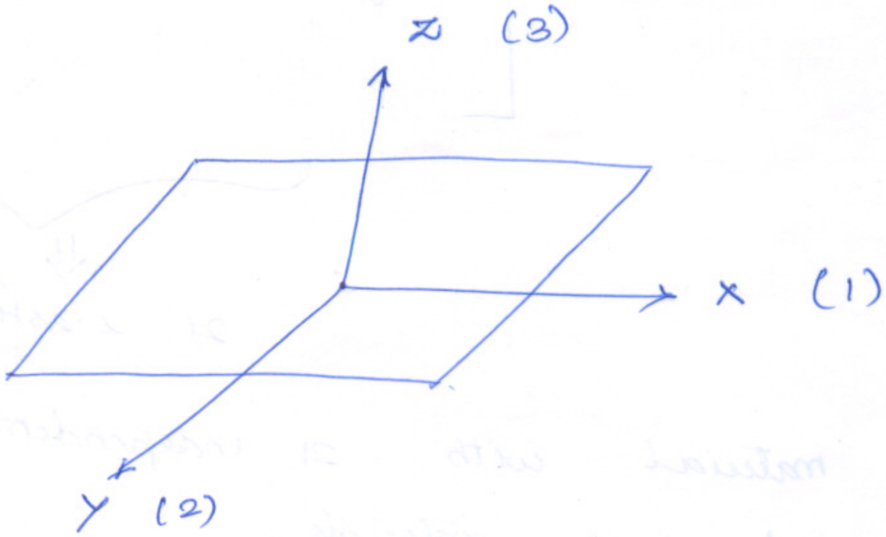
$\therefore$ for $i,j = 1,2 \ldots 6$

$$W = \frac{1}{2} \begin{bmatrix} C_{11}\epsilon_1^2 + 2C_{12}\epsilon_1\epsilon_2 + 2C_{13}\epsilon_1\epsilon_3 + 2C_{14}\epsilon_1\epsilon_4 + 2C_{15}\epsilon_1\epsilon_5 + 2C_{16}\epsilon_1\epsilon_6 + \\ C_{22}\epsilon_2^2 + 2C_{23}\epsilon_2\epsilon_3 + 2C_{24}\epsilon_2\epsilon_4 + 2C_{25}\epsilon_2\epsilon_5 + 2C_{26}\epsilon_2\epsilon_6 + \\ C_{33}\epsilon_3^2 + 2C_{34}\epsilon_3\epsilon_4 + 2C_{35}\epsilon_3\epsilon_5 + 2C_{36}\epsilon_3\epsilon_6 + \\ C_{44}\epsilon_4^2 + 2C_{45}\epsilon_4\epsilon_5 + 2C_{46}\epsilon_4\epsilon_6 + \\ C_{55}\epsilon_5^2 + 2C_{56}\epsilon_5\epsilon_6 + \\ C_{66}\epsilon_6^2 \end{bmatrix} \quad \text{— (3)}$$

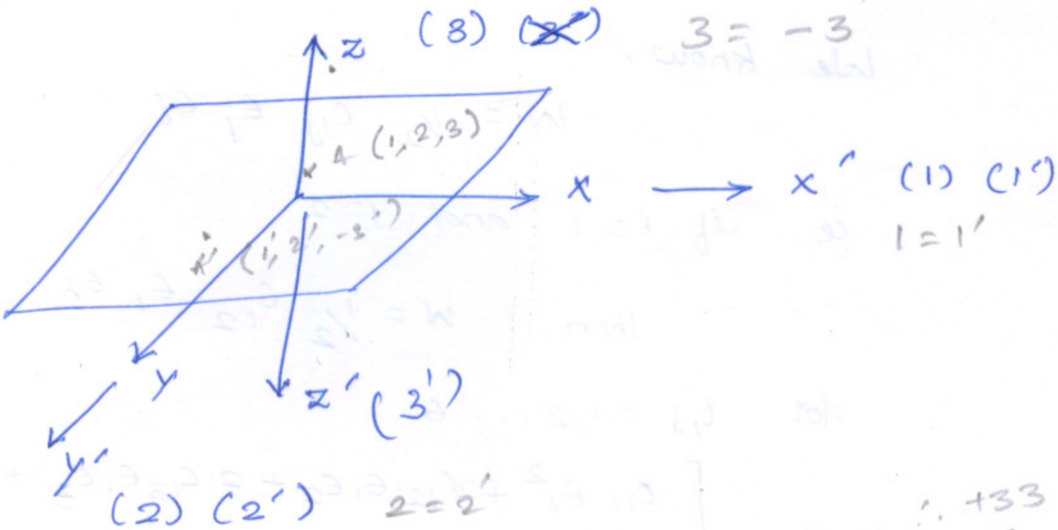
a) Symmetry with respect to a plane.

Consider one plane of material symmetry. A material with one plane of material symmetry is called as monoclinic Material.

Assume the plane of symmetry as $\underline{z=0}$ (or) XY plane.



Since plane of symmetry is XY,



$(-3)(-3)$ $\therefore +33$

$$\Rightarrow \epsilon_{11}' = \epsilon_{11} \;;\; \epsilon_{22}' = \epsilon_{22} \;;\; \epsilon_{33}' = \epsilon_{33}$$

$$\epsilon_{12}' = \epsilon_{12} \;;\; \epsilon_{32}' = -\epsilon_{32} \text{ (or) } \epsilon_{23}' = -\epsilon_{23}$$

$\overset{\epsilon}{(-3)}(+2)$ $\quad -\epsilon_{32}$ $\quad +\epsilon_2(-\epsilon_3)$ $\quad -\epsilon_{23}$

$$\epsilon_{13}' = -\epsilon_{13}$$

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By Applying (4) in (3) we get,

~~Also~~ we know from contracted notation,

~~WKT~~

$$\epsilon_{13} = \epsilon_4 \; ; \quad \epsilon_{23} = \epsilon_5 \; ; \quad \epsilon_{12} = \epsilon_6$$

here, $\epsilon_4' = -\epsilon_4$; $\epsilon_5' = -\epsilon_5$; $\epsilon_6' = \epsilon_6$

$\therefore$

$$W = \frac{1}{2}\Big[c_{11}\epsilon_1^2 + 2c_{12}\epsilon_1\epsilon_2 + 2c_{13}\epsilon_1\epsilon_3 + \cancelto{0}{2c_{14}\epsilon_1\epsilon_4} + \cancelto{0}{2c_{15}\epsilon_1\epsilon_5} + 2c_{16}\epsilon_1\epsilon_6 +$$

$$c_{22}\epsilon_2^2 + 2c_{23}\epsilon_2\epsilon_3 + \cancelto{0}{2c_{24}\epsilon_2\epsilon_4} + \cancelto{0}{2c_{25}\epsilon_2\epsilon_5} + 2c_{26}\epsilon_2\epsilon_6 +$$

$$c_{33}\epsilon_3^2 + \cancelto{0}{2c_{34}\epsilon_3\epsilon_4} + \cancelto{0}{2c_{35}\epsilon_3\epsilon_5} + 2c_{36}\epsilon_3\epsilon_6 +$$

$$c_{44}\epsilon_4^2 + 2c_{45}\epsilon_4\epsilon_5 + \cancelto{0}{2c_{46}\epsilon_4\epsilon_6} +$$

$$c_{55}\epsilon_5^2 + \cancelto{0}{2c_{56}\epsilon_5\epsilon_6} +$$

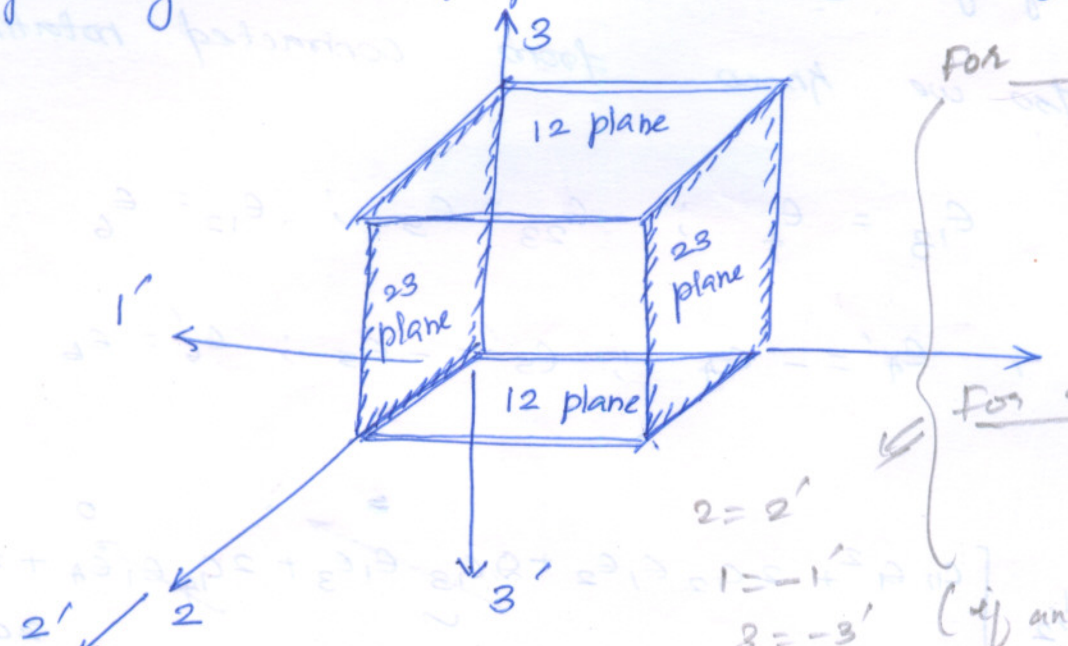
$$c_{66}\epsilon_6^2 \Big] \quad \text{— (5)}$$

$$\therefore \quad c_{ij} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \Rightarrow 13$$

elastic constants.

Monoclinic Material.

b) Symmetry with respect to two orthogonal planes:



For 12 plane

$1 = 1'$

$2 = 2'$

$3 = -3'$

For 23 plane

$2 = 2'$

$3 = 3'$

$1 = -1'$

$2 = 2'$

$1 = -1'$

$3 = -3'$

(if any 1 negative exists it should be considered for resultant)

In addition to 12 plane, Assume 23 plane to be symmetry.

$$\epsilon_1 = \epsilon_{11}' = \epsilon_{11}, \quad \epsilon_{22}' = \epsilon_{22} = \epsilon_2 \; ; \quad \epsilon_{33}' = \epsilon_{33} = \epsilon_3$$

$$\epsilon_{31}' = +\epsilon_{31} = +\epsilon_5, \quad \epsilon_{23}' = -\epsilon_{23} = -\epsilon_4, \quad \epsilon_{12}' = -\epsilon_{12} = -\epsilon_6$$

∴ By substituting the above condition in eqn ⑤, we get,

$$W = \Big[C_{11}\epsilon_1^2 + 2C_{12}\epsilon_1\epsilon_2 + 2C_{13}\epsilon_1\epsilon_3 + \cancel{2C_{16}\epsilon_1\epsilon_6}^{\,0} +$$
$$C_{22}\epsilon_2^2 + 2C_{23}\epsilon_2\epsilon_3 + \cancel{2C_{26}\epsilon_2\epsilon_6}^{\,0} +$$
$$C_{33}\epsilon_3^2 + \cancel{2C_{36}\epsilon_3\epsilon_6}^{\,0} +$$
$$C_{44}\epsilon_4^2 + \cancel{2C_{45}\epsilon_4\epsilon_5}^{\,0} +$$
$$C_{55}\epsilon_5^2 + $$
$$C_{66}\epsilon_6^2 \Big]$$

$$W = \left[C_{11}\epsilon_1^2 + 2C_{12}\epsilon_1\epsilon_2 + 2C_{13}\epsilon_1\epsilon_3 + C_{22}\epsilon_2^2 + 2C_{23}\epsilon_2\epsilon_3 + C_{33}\epsilon_3^2 + C_{44}\epsilon_4^2 + C_{55}\epsilon_5^2 + C_{66}\epsilon_6^2 \right] \quad — (6)$$

$$\Rightarrow \quad C_{ij} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}}_{\Downarrow}$$

9 elastic constants.

The Material with two orthogonal planes of material symmetry is called as orthotropic materials.

NB: If any two orthogonal planes are planes of material symmetry the third mutually orthogonal plane has to be plane to material symmetry.

c) Transverse Isotropy :.

If at every point of a material there is one plane in which the mechanical properties are equal in all directions, then the material is called as transverse isotropic.

Assume 12 plane as plane of isotropy, then the subscripts (1,2) on the stiffness are interchangeable.

i.e, $C_{11} = C_{22}$ ||| r/y $C_{44} = C_{55}$

and $C_{66} = \frac{1}{2}(C_{11} - C_{12})$.

$$\therefore \quad C_{ij} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12}) \end{bmatrix}}$$

$\Downarrow$

5 elastic constants.

d) Isotrophy :-

If there is an infinite number of planes of material property symmetry, then the material is called as isotropic material.

then subscripts 1, 2, 3 are interchangeable

$$\therefore C_{ij} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(C_{11}-C_{12})}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(C_{11}-C_{12})}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(C_{11}-C_{12})}{2} \end{bmatrix}}_{\Downarrow\ \text{2 elastic constants.}}$$

$\sigma - \epsilon$ relation.

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{43} & C_{53} & C_{63} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{54} & C_{64} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{65} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}$$

Shear co-eff/.

$\epsilon_4 = \gamma_{23}$, $\epsilon_5 = \gamma_{31}$, $\epsilon_6 = \gamma_{12}$

$\sigma_4 = \tau_{23}$, $\sigma_5 = \tau_{31}$, $\sigma_6 = \tau_{12}$

$C_{11}, C_{22}, C_{33} \Rightarrow$ Extension co-eff (stress, strain in same normal direction)

12, 13, 23 $\Rightarrow$ Extension - extension coupling co-eff (in different direct)

45, 46, 56 $\Rightarrow$ Shear - shear coupling co-eff. (shear stress, shear strain in diff/. direction).

14, 15, 16, 24, 25, 26, 34, 35, 36 $\Rightarrow$ shear extension coupling (normal stress with shear strain (or) Shear stress with normal strain).